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Moduli space of unstable D-branes on a circle of critical radius

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Abstract: We study the moduli space of the boundary conformal field theories describing an unstable D-brane of type-II string theory compactified on a circle of critical radius. This moduli space has two branches — a three dimensional branch S^3/Z_2 and a two dimensional branch described by a square torus T^2. These two branches are joined along a circle. We compare this with the moduli space of classical solutions of tachyon effective field theory compactified on a circle of critical radius. This moduli space has a very similar structure to that of the boundary conformal field theory with the only difference that the S^3 of the S^3/Z_2 component becomes a deformed S^3. This provides one more indication that the tachyon effective field theory captures qualitatively the dynamics of the tachyon on an unstable D-brane.

Keywords: Tachyon Condensation, D-branes.

Contents

1. Introduction

Classical dynamics of unstable Dp-branes in bosonic and superstring theories have been investigated using various techniques. A particularly interesting configuration where many analytical results can be obtained is when one of the directions of the D-brane is wrapped on a circle of certain critical radius so that the first non-zero momentum mode of the tachyon field along the circle becomes a massless mode in the effective $(p-1,1)$ dimensional theory. The result of switching on classical background value of this massless mode can be represented as a marginal deformation of the boundary conformal field theory (BCFT) describing the dynamics of the D-brane system [1]–[5]. This allows us to derive analytical results about the system at this point. This system is also of physical interest since by an inverse Wick rotation of the coordinate along the circle we can transform this configuration to a time dependent configuration describing the rolling of the tachyon field away from the maximum of the potential [6]–[9].

Since on a circle of critical radius the first momentum mode of the tachyon becomes a marginal deformation of the conformal field theory, we have a family of BCFT's labelled by the classical vacuum expectation value of this mode. There are two massless modes, associated with the two different signs of the momentum along S^1. The vacuum expectation value of these two modes, together with the Wilson line of the U(1) gauge field on the D-brane along the circle, gives a three parameter family of BCFT's. The associated three dimensional moduli space in the case of bosonic string theory is known to be the SU(2) group manifold S^3 [2].

In this paper we analyze the moduli space of the conformal field theory describing an unstable D-brane of superstring theory on a critical circle. The structure of the moduli space turns out to be more complicated, having two branches. One of the branches, on which the original Dp-brane BCFT lies, is a Z_2 orbifold of S^3 [5], with fixed points lying along an equator of S^3. Physically a BCFT associated with a fixed point of the Z_2 transformation represents a D-$(p-1)$ $\bar{\text{D}}$-$(p-1)$ brane pair, transverse to S^1, situated at diametrically opposite points on the circle. From these fixed points emerges a new branch of the moduli space representing D-$(p-1)$-brane $\bar{\text{D}}$-$(p-1)$-brane pair placed at *arbitrary* points on the circle. This new branch has the shape of a square torus, labelled by the positions of the D-$(p-1)$-brane and the $\bar{\text{D}}$-$(p-1)$-brane on the circle S^1. The two branches are joined along a circle, with the circle running along the line of orbifold fixed points in the first branch and along the diagonal of the square torus in the second branch.

Given that the tachyon field has mass^2 of order $-(\alpha')^{-1}$, we clearly cannot hope to understand the complete tachyon dynamics using an effective field theory. Nevertheless it has been found that there is a candidate effective action that describes many features of the dynamics of the tachyon qualitatively [11, 12, 13, 14]. In particular this action has the property that if we compactify one of the directions on a circle of critical radius, then there is a family of classical solutions involving switching on vacuum expectation values of the tachyon and other fields. We analyze this family of classical solutions and determine the moduli space of the solutions. It turns out that the moduli space of the classical solutions has a structure very similar to that of the BCFT's. In particular it has two branches. One of them can be thought of as a Z_2 orbifold of a *deformed* S^3, with a circle of orbifold fixed points. The other one is exactly a square torus as in the case of BCFT. These two branches are again joined along a circle, with the circle running along the fixed point of the Z_2 transformation on the deformed S^3 and along the diagonal of the square torus. This is one more indication that the tachyon effective action of refs. [11]–[14] is capable of giving us qualitative understanding of the full tachyon dynamics in string theory.

The rest of the paper is organized as follows. In section 2 we review the construction of the moduli space of BCFT's describing a D-brane of bosonic string theory on a critical circle. In section 3 we generalize this construction to D-branes in superstring theory. Section 4 is devoted to analyzing the moduli space of classical solutions in the tachyon effective field theory wrapped on a critical circle. In section 5 we discuss how the extra moduli required for the opening up of the other branch T^2 arises in the effective field theory as a result of the zero momentum tachyon mode becoming massless. Throughout the paper we shall use $\alpha' = 1$ unit unless mentioned otherwise.

2. D-string in bosonic string theory on a circle of unit radius

In this section we review, following [2, 5] (see also [1]), the moduli space of the D-brane of bosonic string theory wrapped on a circle of self-dual radius. For definiteness we shall focus our attention on a D-string wrapped on S^1, and ignore the moduli associated with the motion of the D-string in the non-compact transverse directions, but the results are clearly valid for all Dp-branes. If we denote by x the coordinate along the compact direction, then the BCFT associated with the x coordinate admits a three parameter family of marginal deformations, generated by the operators $i\partial X$, $\cos X$ and $\sin X$ respectively. The structure of the moduli space generated by these marginal deformations is best understood by using the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ current algebra that exists at the self-dual radius. The SU(2) currents are given by:

$$\begin{aligned} J_L^3 &\sim i\partial X_L\,, \qquad & J_L^1 &\sim \cos(2X_L)\,, \qquad & J_L^2 &\sim \sin(2X_L)\,, \\ J_R^3 &\sim i\partial X_R\,, \qquad & J_R^1 &\sim \cos(2X_R)\,, \qquad & J_R^2 &\sim \sin(2X_R)\,, \end{aligned} \tag{2.1}$$

where X_L and X_R denotes the left and the right-moving components of X so that $X = X_L + X_R$. The boundary condition on X is given by

$$X_L = X_R\,, \tag{2.2}$$

which gives

$$J_L^a = J_R^a\,, \qquad \text{for } 1 \le a \le 3\,, \tag{2.3}$$

at the boundary. This shows that the boundary operators $i\partial X$, $\cos X$ and $\sin X$ can be regarded as restrictions of J_L^3, J_L^1 and J_L^2 (or J_R^3, J_R^1 and J_R^2) respectively to the boundary.

Since J^a's generate SU(2) transformation, the moduli space of the BCFT associated with the boundary deformations generated by $i\partial X$, $\cos X$ and $\sin X$ is simply the SU(2) group manifold S^3 [2, 5]. We shall parametrize this manifold by choosing the following representation of the SU(2) group element:

$$g = \exp\left(\frac{1}{2} i\chi\sigma_3\right) \exp\left(\frac{1}{2} i\phi\sigma_3\right) \exp\left(i\theta\sigma_1\right) \exp\left(-\frac{1}{2} i\phi\sigma_3\right) \exp\left(\frac{1}{2} i\chi\sigma_3\right), \tag{2.4}$$

where σ_1, σ_2, σ_3 are Pauli matrices. In this parametrization the metric on S^3 is given by:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\chi^2\,. \tag{2.5}$$

Each of θ, ϕ and χ can be taken to be angular variables with periodicity 2π, provided we make the following additional identifications

$$(\theta, \phi, \chi) \equiv (2\pi - \theta, \phi + \pi, \chi)\,, \qquad (\theta, \phi, \chi) \equiv (\pi - \theta, \phi, \chi + \pi)\,. \tag{2.6}$$

Thus we can take the fundamental region of the moduli space to be:

$$0 \le \theta \le \frac{\pi}{2}\,, \qquad 0 \le \phi \le 2\pi\,, \qquad 0 \le \chi \le 2\pi\,. \tag{2.7}$$

The S^3 nature of the manifold can be made manifest by associating to (θ, ϕ, χ) the point $(\cos\theta\cos\chi, \cos\theta\sin\chi, \sin\theta\cos\phi, \sin\theta\sin\phi)$ in R^4. These points clearly span a unit three sphere in R^4.

It is instructive to try to identify the parameters (θ, ϕ, χ) with the explicit boundary deformations generated by $i\partial X$, $\cos X$ and $\sin X$. For this we begin with the observation that a generic boundary perturbation of the form $-i\varphi^a \oint J_L^a$ for some parameters φ^a, with $\oint J_L^a$ denoting $(2\pi i)^{-1}$ times an integral of $J_L^a(z)$ over z along the boundary of the world-sheet, corresponds to an insertion of $\exp(i\varphi^a \oint J_L^a)$ in the euclidean world-sheet path integral. (The choice of $\mathrm{SU}(2)_L$ is purely a matter of convention; we could have also chosen the $\mathrm{SU}(2)_R$ group since the boundary condition $J_L^a = J_R^a$ relates the two sets of generators.)

Thus any BCFT associated with deformation by the operators $i\partial X$, $\cos X$ and $\sin X$ can be associated with an $\mathrm{SU}(2)_L$ group element $\exp(i\varphi^a \oint J_L^a)$. We can now make a natural association between the SU(2) group element g given in (2.4) and an $\mathrm{SU}(2)_L$ group element by associating $\frac{1}{2}\sigma_a$ with the generators $\oint J_L^a$. Thus the element g given in (2.4) is associated with the following insertion in the path integral:

$$\exp\left(i\chi \oint J_L^3\right) \exp\left(i\phi \oint J_L^3\right) \exp\left(2i\theta \oint J_L^1\right) \exp\left(-i\phi \oint J_L^3\right) \exp\left(i\chi \oint J_L^3\right). \quad (2.8)$$

This can be interpreted as the insertion of a series of exponentiated contour integrals, with the left-most element containing the outermost contour closest to the world-sheet boundary and the rightmost element containing the innermost contour. Now the $\oint J_L^3$ in the left-most contour can be converted into a $-\oint J_R^3$ using the boundary condition (2.3), the change in sign being due to the change in the direction of integration. Since J_R^a commute with J_L^b, this element can now be taken all the way to the right and combined with the last term to give $\exp(i\chi \oint (J_L^3 - J_R^3))$. In this new configuration $\exp(i\phi \oint J_L^3)$ becomes the left-most term, and hence again the $\oint J_L^3$ in this term can be converted into a $-\oint J_R^3$ using the boundary condition (2.3). Again using the fact that J_R^a commutes with J_L^b, we can bring this term to the right of $\exp(2i\theta \oint J_L^1)$ and combine it with the $\exp(-i\phi \oint J_L^3)$ term. Thus the net result is:

$$\exp\left(2i\theta \oint J_L^1\right) \exp\left(-i\phi \oint (J_L^3 + J_R^3)\right) \exp\left(i\chi \oint (J_L^3 - J_R^3)\right). \quad (2.9)$$

This has the following interpretation. The insertion of $\exp(2i\theta \oint J_L^1)$ corresponds to deforming the BCFT by a term proportional to $\cos X$. The insertion of $\exp(-i\phi \oint (J_L^3 + J_R^3))$ corresponds to translating the world-sheet field X by an amount proportional to ϕ, since $\oint (J_L^3 + J_R^3)$ is the generator of ordinary translation. This can also be seen explicitly by rewriting the piece $\exp(i\phi \oint J_L^3) \exp(2i\theta \oint J_L^1) \exp(-i\phi \oint J_L^3)$ in the original expression (2.8) as $\exp(2i\theta \oint (J_L^1 \cos\phi - J_L^2 \sin\phi))$. This, using (2.1), corresponds to a boundary deformation proportional to $\cos(X + \phi)$. Thus the effect of ϕ is to essentially convert the $\cos X$ perturbation to $\cos(X + \phi)$. Finally, the effect of $\exp(i\chi \oint (J_L^3 - J_R^3))$ is to induce a translation proportional to χ along the T-dual circle, since $\oint (J_L^3 - J_R^3)$ is the generator of translation along the T-dual circle. Since the operators $\oint (J_L^3 \pm J_R^3)$ multiplying ϕ and χ are normalized in the same way, and since ϕ is precisely the amount of translation on the original circle, χ must measure precisely the amount of translation along S_D^1. This is also consistent with the fact that χ appearing in (2.4), as well as the T-dual circle S_D^1, has periodicity 2π. In the original description switching on χ corresponds to inducing a Wilson line χ along S^1, normalized so as to have periodicity 2π.

For later use, we would like to find the constant of proportionality between the parameter θ and the coefficient $\widetilde{\lambda}$ of the $\cos X$ boundary perturbation that appears e.g. in [6, 7]. For this we note that at $\theta = 0$ the orbits generated by ϕ translation shrink to points. Since ϕ represents translation along S^1, this implies that the $\theta = 0$ configuration is invariant under translation along S^1. This is of course consistent with the identification of the $\theta = 0$ point with a D-string wrapped on S^1 which is translationally invariant. We also see from (2.5) that at $\theta = \pi/2$ the orbits generated by χ translation shrink to points. Since χ-translation corresponds to translation along the dual circle S^1_D, this shows that the $\theta = \pi/2$ point must represent a configuration of the D-brane which is invariant under translation along the dual circle. Thus it is natural to identify the $\theta = \pi/2$ point as the point where the original D-string is converted to a D0-brane, *i.e.* the $\widetilde{\lambda} = \frac{1}{2}$ point in the notation of [6]. Since T-duality converts this to a D-string wrapped on a dual circle, this configuration is natuarally invariant under translation along the dual circle. Thus the parameter θ is related to the parameter $\widetilde{\lambda}$ of [6] through the relation:

$$\theta = \pi\widetilde{\lambda}\,. \tag{2.10}$$

To summarize, in order to associate a BCFT corresponding to a given point (θ, ϕ, χ) on the SU(2) group manifold, we proceed as follows. First we switch on a boundary deformation proportional to $\theta\cos(X + \phi)$ with the overall normalization adjusted so that $\theta = \pi/2$ corresponds to the 'Dirichlet point'. We then go to the T-dual description and translate the BCFT by an amount χ on the T-dual circle. Finally we go back to the original description by making a further T-duality transformation.

Let us now check that the metric (2.5) on the SU(2) group manifold agrees with the metric on the moduli space of classical solutions in open string field theory for small θ. For this we consider the quadratic part of the open string field theory action describing small fluctuations of the tachyon field T and the gauge field A_x on the D-string wrapped on S^1:

$$S^{(2)} = -\mathcal{T}_1 \int dt \int_0^{2\pi} dx \left[1 - \frac{1}{2}T^2 - \frac{1}{2}\dot{T}^2 + \frac{1}{2}(T')^2 - \frac{1}{2}(\dot{A}_x)^2\right], \tag{2.11}$$

where $\mathcal{T}_1$ denotes the tension of the D-string. The constant term in the action denotes the contribution due the tension of the D-string. The gauge field component A_x is an angular variable, since it measures the Wilson line along S^1. To determine the periodicity of A_x, we begin with the observation that for $T = 0$, $A_x = 0$, the total energy of the brane is $2\pi\mathcal{T}_1$. In the T-dual description this can be interpreted as the mass $\mathcal{T}_0$ of the D0-brane on S^1_D. We now note that for x independent A_x, the term involving $\dot{A}_x$ in (2.11) has the form

$$\frac{1}{2}\mathcal{T}_0 \int dt\, (\dot{A}_x)^2\,. \tag{2.12}$$

This can be interpreted as the action for the D0-brane moving on S^1_D if we interprete A_x as the position of the D0-brane on S^1_D. Since S^1_D has unit radius, we see that A_x is an angular coordinate with periodicity 2π.

Consider now a classical soliton solution of this field theory given by

$$T = \lambda\cos x\,. \tag{2.13}$$

The massless fluctuations around this solution are associated with a general field configuration of the form:

$$T(x,t) = \lambda(t)\cos(x + \alpha(t))\,, \qquad A_x(x,t) = \beta(t)\,, \tag{2.14}$$

where λ, α and β are collective coordinates. From (2.14) it is clear that α is an angular variable with periodicity 2π. Since A_x is an angular variable with periodicity 2π, it also follows from (2.14) that β is an angular variable with periodicity 2π. Substituting (2.14) into the action (2.11) we get the action for the collective coordinates:

$$S^{(2)} = -\mathcal{T}_0 \int dt \left[1 - \frac{1}{4}\dot{\lambda}^2 - \frac{1}{4}\lambda^2\dot{\alpha}^2 - \frac{1}{2}\dot{\beta}^2\right] . \tag{2.15}$$

We want to identify the collective coordinates λ, α and β with the coordinates θ, ϕ and χ labelling the SU(2) moduli space near $\theta = 0$. Since α generates translation along S^1, we have $\alpha = \phi$. β, being the Wilson line along S^1, generates translation along the dual circle S^1_D. Thus β must be proportional to χ. Since both χ and β have periodicity 2π, we have $\beta = \chi$. Finally we need to find the relation between the parameter λ labelling the solution in the open string field theory and the parameter θ (or equivalently the parameter $\widetilde{\lambda} = \theta/\pi$) labelling the solution in the BCFT. For this we compare the xx component of the stress tensor in the two descriptions.[1] In the open string field theory we have for the solution (2.13),

$$T_{xx} = -\mathcal{T}_1 \left(1 - \frac{1}{2}(T')^2 - \frac{1}{2}T^2\right) = -\mathcal{T}_1 \left(1 - \frac{1}{2}\lambda^2\right) . \tag{2.16}$$

On the other hand for the BCFT the T_{xx} associated with the boundary deformation $\widetilde{\lambda}\cos(X)$ is obtained by Wick rotation of the answer for T_{00} associated with the boundary deformation $\widetilde{\lambda}\cosh(X^0)$. This gives [7]:

$$T_{xx} = -\mathcal{T}_1 \cos^2(\pi\widetilde{\lambda}) = -\mathcal{T}_1 \cos^2\theta \simeq -\mathcal{T}_1(1 - \theta^2) , \tag{2.17}$$

for small $\widetilde{\lambda}$ i.e. small θ. Comparing (2.16) and (2.17) we see that for small λ we must have the identification $\lambda \simeq \sqrt{2}\theta$. Thus all in all, we have:

$$\alpha = \phi , \qquad \beta = \chi , \qquad \lambda \simeq \sqrt{2}\theta , \tag{2.18}$$

Substituting this into (2.15) we get

$$S^{(2)} \simeq -\mathcal{T}_0 \int dt \left[1 - \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}\theta^2\dot{\phi}^2 - \frac{1}{2}\dot{\chi}^2\right] . \tag{2.19}$$

This corresponds to a moduli space metric:

$$ds^2 \simeq (d\theta^2 + \theta^2 d\phi^2 + d\chi^2) , \tag{2.20}$$

near $\theta = 0$. This agrees with the exact metric (2.5) near $\theta = 0$.

[1] Since T_{xx} acts as the source for the zero momentum closed string sector scalar field g_{xx}, this is a physical quantity. Thus it is appropriate to match the parameters in the two descriptions by comparing the T_{xx} for the two systems.

3. Unstable D-string of superstring theory on a circle of radius $\sqrt{2}$

In this section we shall repeat the analysis of the previous section for an unstable D-brane in superstring theory, wrapped on a circle S^1 of radius $\sqrt{2}$, and show that it has the structure of S^3/Z_2 [5] and T^2 joined together along a circle. Again for definiteness we shall focus on the case of unstable D-string of type-IIA string theory and ignore the moduli associated with the transverse motion of the D-string, but the results are valid for any unstable D-brane in type-IIA or IIB string theory. Let x denote the coordinate along the circle, X be the associated world-sheet scalar field, and ψ be the world-sheet superpartner of X. At the critical radius $\sqrt{2}$ the BCFT associated with the fields (X, ψ) admits three marginal boundary deformations [3, 4].[2] Up to Chan-Paton factors, these operators are

$$i\partial X\,, \qquad \psi \sin\left(\frac{X}{\sqrt{2}}\right), \qquad -\psi\cos\left(\frac{X}{\sqrt{2}}\right). \tag{3.1}$$

Here we have used a zero picture representation [15]. These represent respectively the effect of switching on a Wilson line along S^1, switching on a tachyon background proportional to $\cos(x/\sqrt{2})$ and a tachyon background proportional to $\sin(x/\sqrt{2})$. In order to analyze the BCFT's generated by these marginal deformations, we use a fermionic representation for the bosonic field X. We have, up to cocycle and numerical factors [3, 4]:

$$e^{\pm i\sqrt{2}X_L} \sim (\xi_L \pm i\eta_L)\,, \qquad e^{\pm i\sqrt{2}X_R} \sim (\xi_R \pm i\eta_R)\,, \qquad i\partial X_L \sim \eta_L\xi_L\,, \qquad i\partial X_R \sim \eta_R\xi_R\,, \tag{3.2}$$

where (ξ_L, η_L) denote a pair of left-handed Majorana-Weyl fermion and (ξ_R, η_R) denote a pair of right-handed Majorana-Weyl fermion. The Neumann boundary condition $X_L = X_R$, $\psi_L = \psi_R$ translates to:

$$\xi_L = \xi_R \equiv \xi\,, \qquad \eta_L = \eta_R \equiv \eta\,, \qquad \psi_L = \psi_R \equiv \psi\,. \tag{3.3}$$

Thus the conformal field theory described by (X, ψ) can be equally well described as a theory of three free fermion fields ψ, ξ and η. This theory has an underlying level two $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ current algebra, generated by:

$$J^1_L \sim \psi_L\eta_L\,, \quad J^2_L \sim \xi_L\psi_L\,, \quad J^3_L \sim \eta_L\xi_L\,, \quad J^1_R \sim \psi_R\eta_R\,, \quad J^2_R \sim \xi_R\psi_R\,, \quad J^3_R \sim \eta_R\xi_R\,. \tag{3.4}$$

The boundary condition (3.3) gives

$$J^a_L = J^a_R \tag{3.5}$$

on the boundary. The existence of this current algebra does not however mean that the underlying theory is invariant under this symmetry. For example, in the closed string sector the GSO projection rules do not commute with the $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ symmetry. In the open string sector the spectrum is invariant under the diagonal SU(2) group, but the correlation functions are not SU(2) invariant since the assignment of Chan Paton and cocycle factors break this SU(2) symmetry [3, 4]. Nevertheless we shall see that the SU(2) symmetry can be exploited to determine the structure of the moduli space.

[2] In the notation used later in this section, [4] describes the marginal deformation of the theory from the $\theta = 0$ point to the $\theta = \pi/2$ point, whereas [3] describes the marginal deformation of the theory from the $\theta = \pi/2$ point to the $\theta = 0$ point.

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Using eqs. (3.2)–(3.5), and that $X = X_L + X_R$, the three boundary operators listed in eq. (3.1) may be expressed as:

$$i\partial X \sim \eta_L \xi_L \sim J_L^3 \,, \qquad \psi \sin\left(\frac{X}{\sqrt{2}}\right) \sim \psi_L \eta_L \sim J_L^1 \,, \qquad -\psi \cos\left(\frac{X}{\sqrt{2}}\right) \sim \xi_L \psi_L \sim J_L^2 \,. \tag{3.6}$$

Had SU(2) been an exact symmetry of the problem, this would again imply that the moduli space of the BCFT generated by these marginal deformations is locally SU(2) group manifold S^3. As we have pointed out, SU(2) is not a symmetry of the full string theory in this case. However for the marginal operators listed in (3.6), the corresponding Chan-Paton times cocycle factors commute (being identity for the first operator and the Pauli matrix Σ_1 for the second and the third operator [4]), and hence plays no role in the computation of the correlation function of these operators. Since the computation of the moduli space metric only involves computing correlation functions of these operators (together with inverse picture changing operators [16] which are manifestly SU(2) invariant), we see that as far as computation of the moduli space metric is concerned, we can treat the SU(2) symmetry to be unbroken. The marginal deformations generated by the boundary values of J_L^a $(1 \le a \le 3)$ then generate the SU(2) group manifold S^3, at least locally. As we shall see shortly, global structure of the moduli space differs from that of the SU(2) group manifold in a subtle way [5].

As in the previous section, we shall label the SU(2) group element as

$$g = \exp\left(\frac{1}{2} i\chi\sigma_3\right) \exp\left(\frac{1}{2} i\phi\sigma_3\right) \exp\left(i\theta\sigma_1\right) \exp\left(-\frac{1}{2} i\phi\sigma_3\right) \exp\left(\frac{1}{2} i\chi\sigma_3\right), \tag{3.7}$$

so that the metric on S^3 takes the form:

$$ds^2 = 2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\chi^2) \,. \tag{3.8}$$

The overall normalization factor of 2 reflects that we have a level two current algebra. We can associate a specific BCFT to the group element g given in (3.7) by following the same procedure as in the case of bosonic string theory. As before, the effect of switching on θ is to switch on a marginal boundary deformation proportional to $J_L^1 \sim \psi \sin(X/\sqrt{2})$, which in turn represents a tachyon background proportional to $\cos(X/\sqrt{2})$. Up to an overall normalization ϕ and χ represent respectively the effect of translating the BCFT along the original circle S^1 of radius $\sqrt{2}$ and along the dual circle S_D^1 of radius $1/\sqrt{2}$. In particular, the effect of switching on ϕ is to rotate σ_1 to $\sigma_1 \cos\phi - \sigma_2 \sin\phi$, and hence J^1 to $J^1 \cos\phi - J^2 \sin\phi \sim \psi \sin(\frac{X}{\sqrt{2}} + \phi)$. This corresponds to translation of X by $\sqrt{2}\phi$. From this we can conclude that χ will induce a translation along S_D^1 by an amount $\sqrt{2}\chi$ since the operators multiplying χ and ϕ — $\oint (J_L^3 \pm J_R^3)$ — are normalized in the same way. Finally, the relative normalization between θ and the parameter $\widetilde{\lambda}$ used in [7] is determined as follows. The collapse of the χ orbit at $\theta = \pi/2$ implies that the corresponding configuration must be invariant under translation along the dual circle S_D^1. Thus this must correspond to the point $\widetilde{\lambda} = 1/2$ in the notation of ref. [7] where the tachyon background converts the D1-brane into a D0-$\bar{\text{D}}$0-brane pair situated at the diametrically opposite points of the circle.

Under T-duality this maps to a D1- $\bar{\text{D}}$1-brane pair wrapped on S^1_D with half unit of Wilson line on one of them, and hence is invariant under translation along S^1_D. This gives

$$\theta = \pi\widetilde{\lambda}\,. \tag{3.9}$$

Since S^1_D has radius $1/\sqrt{2}$, and χ induces a translation along S^1_D by $\sqrt{2}\chi$, we see that χ has periodicity π. On the other hand, χ labelling the SU(2) group manifold S^3 has periodicity 2π. This shows that the moduli space of BCFT's, instead of being an SU(2) group manifold S^3, is actually a Z_2 quotient of S^3 by the transformation $\chi \to \chi + \pi$ [5]. We can see the physical origin of the identification under $\chi \to \chi + \pi$ by working near the point $\theta = \pi/2$, i.e. $\widetilde{\lambda} = 1/2$. This configuration represents a D0-$\bar{\text{D}}$0-brane pair at diametrically opposite points on the circle S^1 [3, 4]. In the T-dual version we have a D1-$\bar{\text{D}}$1-brane pair on the dual circle S^1_D of radius $1/\sqrt{2}$, with half a unit of Wilson line along one of the D-branes. As a result the tachyonic modes in the open string sector with one end on the D-string and the other end on the $\bar{\text{D}}$-string are forced to carry half-integral units of momentum $(n+\frac{1}{2})\sqrt{2}$. The lowest modes, carrying momenta $\pm\frac{1}{\sqrt{2}}$, are massless. Deforming θ away from $\pi/2$ corresponds to switching on vacuum expectation values of some of these modes of the form $(\frac{1}{2}-\widetilde{\lambda})\cos(\frac{x_D}{\sqrt{2}}+\chi) \sim (\frac{\pi}{2}-\theta)\cos(\frac{x_D}{\sqrt{2}}+\chi)$, x_D being the coordinate labelling the dual circle. Since S^1_D has radius $1/\sqrt{2}$, any given configuration must go into a physically equivalent configuration under $x_D \to x_D + 2\pi/\sqrt{2}$. However under this translation $(\frac{\pi}{2}-\theta)\cos(\frac{x_D}{\sqrt{2}}+\chi)$ changes sign rather than remaining invariant. This shows that we must identify the pair of solutions $\pm(\frac{\pi}{2}-\theta)\cos(\frac{x_D}{\sqrt{2}}+\chi)$, i.e. identify the points (θ,ϕ,χ) and $(\pi-\theta,\phi,\chi)$, or equivalently, the points (θ,ϕ,χ) and $(\theta,\phi,\chi+\pi)$. The resulting manifold can be parametrized by (θ,ϕ,χ) with the identification:

$$(\theta,\phi,\chi) \equiv (\theta+2\pi,\phi,\chi) \equiv (\theta,\phi+2\pi,\chi) \equiv (\theta,\phi,\chi+\pi) \equiv (2\pi-\theta,\phi+\pi,\chi) \equiv (\pi-\theta,\phi,\chi)\,. \tag{3.10}$$

This, in turn, allows us to choose the fundamental region labelling the moduli space to be:

$$0 \le \theta \le \frac{\pi}{2}\,, \qquad 0 \le \phi \le 2\pi\,, \qquad 0 \le \chi \le \pi\,. \tag{3.11}$$

To summarize the results obtained so far, we have the following identification between a point (θ,ϕ,χ) on S^3/Z_2 and a deformation of the BCFT describing an unstable D-brane. We first switch on a boundary deformation proportional to $\theta\psi\sin(\frac{X}{\sqrt{2}}+\phi)$, normalized so that $\theta = \pi/2$ corresponds to the 'Dirichlet point'. We then go to the T-dual description of this conformal field theory and translate the BCFT by an amount $\sqrt{2}\chi$ along S^1_D. Finally we go back to the original description by making a further T-duality transformation.

We shall now repeat the analysis described at the end of section 2 to check that the moduli space metric (3.8) is compatible with the metric on the moduli space of classical solutions in open superstring field theory for small θ. In this case the quadratic part of the string field theory action describing the dynamics of the tachyon field T and the gauge field A_x on the D-string wrapped on S^1 is given by:

$$S^{(2)} = -\mathcal{T}_1 \int dt \int_0^{2\pi\sqrt{2}} dx \left[1 - \frac{1}{4}T^2 - \frac{1}{2}\dot{T}^2 + \frac{1}{2}(T')^2 - \frac{1}{2}(\dot{A}_x)^2\right], \tag{3.12}$$

where $\mathcal{T}_1$ denotes the tension of the non-BPS D-string. The factor of $1/4$ in front of T^2 reflects that the tachyon on a non-BPS D-brane in superstring theory has mass$^2 = -1/2$. At $T = 0$, $A_x = 0$ the total mass of the system is given by $2\pi\sqrt{2}\mathcal{T}_1$. Under a T-duality the bulk theory gets mapped to type-IIB string theory compactified on a dual circle of radius $1/\sqrt{2}$, and the D-string gets mapped to a D0-brane of type-IIB theory with mass $\mathcal{T}_0 = 2\pi\sqrt{2}\mathcal{T}_1$. Thus the term involving A_x can be written as

$$\frac{1}{2}\mathcal{T}_0 \int dt\, (\dot{A}_x)^2 \,. \tag{3.13}$$

Identifying this with the kinetic term for the D0-brane associated with its motion along S^1_D, we see that A_x denotes the position of the D0-brane on S^1_D. Thus A_x is an angular variable with periodicity $2\pi/\sqrt{2}$.

We now consider a classical soliton solution of this field theory given by

$$T = \lambda \cos\left(\frac{x}{\sqrt{2}}\right) . \tag{3.14}$$

The massless fluctuations around this solution are associated with a general field configuration of the form:

$$T(x,t) = \lambda(t) \cos\left(\frac{x}{\sqrt{2}} + \alpha(t)\right) , \qquad A_x(t) = \frac{\beta(t)}{\sqrt{2}} \,, \tag{3.15}$$

where λ, α and β are collective coordinates. With this normalization, α and β are both angular variables with periodicity 2π. Substituting (3.15) into the action (3.12) we get the action for the collective coordinates:

$$S^{(2)} = -\mathcal{T}_0 \int dt \left[1 - \frac{1}{4}\dot{\lambda}^2 - \frac{1}{4}\lambda^2\dot{\alpha}^2 - \frac{1}{4}\dot{\beta}^2\right] . \tag{3.16}$$

We need to relate the collective coordinates λ, α and β with the coordinates θ, ϕ and χ labelling the SU(2) moduli space near $\theta = 0$. Since both α and ϕ generate translation along S^1 and both have periodicity 2π, we must have $\alpha = \phi$. On the other hand, both β and χ generate translation along the dual circle S^1_D, but β has periodicity 2π whereas χ has periodicity π. Thus we have $\beta = 2\chi$. Finally, to find the relation between the parameter λ and the parameter θ (or equivalently the parameter $\widetilde{\lambda} = \theta/\pi$) labelling the BCFT, we compare the xx component of the stress tensor in the two descriptions. In the string field theory the xx component of the stress tensor associated with the solution $T = \lambda\cos(x/\sqrt{2})$ is

$$T_{xx} = -\mathcal{T}_1 \left(1 - \frac{1}{4}T^2 - \frac{1}{2}(T')^2\right) = -\mathcal{T}_1 \left(1 - \frac{1}{4}\lambda^2\right) . \tag{3.17}$$

On the other hand T_{xx} associated with the BCFT obtained by deforming the world-sheet action by a boundary term proportional to $\widetilde{\lambda}\psi \sinh(X/\sqrt{2})$ is obtained by Wick rotating the answer for T_{00} associated with the boundary deformation $\widetilde{\lambda}\psi^0 \sinh(X^0/\sqrt{2})$. This can be read out from ref. [7] and gives:

$$T_{xx} = -\mathcal{T}_1 \cos^2(\pi\widetilde{\lambda}) = -\mathcal{T}_1 \cos^2\theta \simeq -\mathcal{T}_1(1-\theta^2) \,, \tag{3.18}$$

for small θ. Comparison of (3.17) and (3.18) gives, for small θ, $\lambda \simeq 2\theta$. Thus the complete relation between (λ, α, β) and (θ, ϕ, χ) for small θ is given by:

$$\alpha = \phi\,, \qquad \beta = 2\chi\,, \qquad \lambda \simeq 2\theta\,. \tag{3.19}$$

Substituting this into (3.16) we get

$$S^{(2)} \simeq -\mathcal{T}_0 \int dt\, [1 - \dot\theta^2 - \theta^2\dot\phi^2 - \dot\chi^2]\,. \tag{3.20}$$

This corresponds to the metric:

$$ds^2 \simeq 2(d\theta^2 + \theta^2 d\phi^2 + d\chi^2)\,, \tag{3.21}$$

and agrees with (3.8) near $\theta = 0$.

The above analysis confirms our earlier assertion that the moduli space of BCFT's associated with marginal boundary deformation of an unstable D-brane wrapped on a critical circle is S^3/Z_2. However this is only one branch of the moduli space. At the point $\theta = \pi/2$ there are other branches of the moduli space of the BCFT. To see this we go to the T-dual picture where the $\theta = \pi/2$ point represents a D1-$\bar{\text{D}}$1-brane pair, each wrapped on a dual circle S^1_D, with the $\bar{\text{D}}$1-brane carrying half a unit of Wilson line. This configuration supports [3] six massless modes, associated with the (zero picture) vertex operators:

$$i\partial X_D\,, \qquad i\partial X_D \otimes \sigma_3\,, \qquad \psi_D \sin\frac{X_D}{\sqrt{2}} \otimes \sigma_1\,, \qquad -\psi_D \cos\frac{X_D}{\sqrt{2}} \otimes \sigma_1\,,$$
$$\psi_D \sin\frac{X_D}{\sqrt{2}} \otimes \sigma_2\,, \qquad -\psi_D \cos\frac{X_D}{\sqrt{2}} \otimes \sigma_2\,, \tag{3.22}$$

where the Pauli matrices σ_i label the Chan-Paton factors, and ψ_D and X_D denote the T-dual of ψ and X respectively. The coefficient of $i\partial X_D \otimes \sigma_3$ measures the relative Wilson line between the D1-brane and the $\bar{\text{D}}$1-brane. Switching on this Wilson line makes all the other modes except $i\partial X_D$ massive or tachyonic. Thus there is a two dimensional branch of the moduli space, labelled by the coefficients of the operators $i\partial X_D$ and $i\partial X_D \otimes \sigma_3$, emanating from $\theta = \pi/2$. In the original description a point on this two dimensional branch represents a D0-$\bar{\text{D}}$0 brane pair placed at two arbitrary points on the circle S^1. Thus the moduli space has the structure of a square torus $T^2 = S^1 \times S^1$, with flat metric. This branch has a one dimensional intersection with the earlier branch S^3/Z_2. On the square torus, this one dimensional line runs along the diagonal, representing D0-$\bar{\text{D}}$0-brane pairs at diametrically opposite points on the circle. On S^3/Z_2, this line is the equator labelled by ϕ, situated at $\theta = \pi/2$.

If we set the coefficient of the operator $i\partial X_D \otimes \sigma_3$ to zero, we can switch on vev of the other massless fields. However we cannot simultaneously switch on all of the five other massless fields, since in order to have a mutually marginal set of operators we need to have operators with commuting Chan-Paton factors. This leaves us with a one parameter (τ) family of three dimensional branches, generated by the operators:

$$i\partial X_D\,, \qquad \psi_D \sin\frac{X_D}{\sqrt{2}} \otimes (\cos\tau\sigma_1 + \sin\tau\sigma_2)\,, \qquad -\psi_D \cos\frac{X_D}{\sqrt{2}} \otimes (\cos\tau\sigma_1 + \sin\tau\sigma_2)\,. \tag{3.23}$$

The operators for different values of τ however are related to each other by a U(1) gauge symmetry on the world volume of the D1-$\bar{\text{D}}$1 system, generated by σ_3. Since in constructing the moduli space of solutions we must identify solutions which are related by open-string gauge transformations, we see that there really is just one three dimensional branch. We can take this to be the branch corresponding to $\tau = 0$, i.e. the branch already discussed earlier. This argument also shows that the solutions associated with $\tau = 0$ and $\tau = \pi$ must be identified. This changes the sign of the operators $\psi_D \cos X_D \otimes \sigma_1$ and $\psi_D \sin X_D \otimes \sigma_1$, and precisely corresponds to the $\chi \to \chi + \pi$ transformation discussed earlier. Thus we see that the origin of the Z_2 quotient can be traced back to a residual Z_2 subgroup of the U(1) gauge symmetry on the D1-$\bar{\text{D}}$1 brane system.

To summarize, the moduli space of the BCFT's associated with a non-BPS D-string wrapped on a circle of radius $\sqrt{2}$ has two branches — a three dimensional branch S^3/Z_2, and a two dimensional branch with the geometry of a square torus T^2. These two branches are joined along a circle. On the torus this circle runs along the diagonal of the square torus, whereas on S^3/Z_2 the circle runs along the line of orbifold fixed points situated at $\theta = \pi/2$.

4. Moduli space of solutions in the tachyon effective field theory

Many of the qualitative (and some quantitative) features of the tachyon dynamics on an unstable Dp-brane are described by the tachyon effective action [11]–[14]:

$$\begin{aligned} S &= \int d^{p+1}x\, \mathcal{L}\,, \\ \mathcal{L} &= -V(T)\sqrt{-\det A}\,, \end{aligned} \tag{4.1}$$

where

$$A_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu T \partial_\nu T + \partial_\mu Y^I \partial_\nu Y^I + F_{\mu\nu}\,, \tag{4.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{4.3}$$

A_μ and Y^I for $0 \le \mu, \nu \le p$, $(p+1) \le I \le 9$ are the gauge and the transverse scalar fields on the world-volume of the non-BPS brane, and T is the tachyon field. $V(T)$ is the tachyon potential:

$$V(T) = \frac{\mathcal{T}_p}{\cosh(T/\sqrt{2})}\,. \tag{4.4}$$

We shall focus on the case $p = 1$ and take the coordinate $x^1 = x$ to be along a circle S^1 of radius $\sqrt{2}$. We shall also ignore the dynamics of the transverse coordinates Y^I since they simply represent transverse motion of the brane and can be incorporated at a later stage if needed. In this case the action (4.1) in the $A_0 = 0$ gauge reduces to:

$$S = -\mathcal{T}_1 \int dt \int_0^{2\pi\sqrt{2}} dx \frac{1}{\cosh(T/\sqrt{2})} \sqrt{1 - \dot{T}^2 + (T')^2 - (\dot{A}_x)^2}\,. \tag{4.5}$$

Since for small T and A_x the action (4.1) reduces to the quadratic action (3.12), the arguments of section (3) can be used to conclude that A_x has periodicity $2\pi/\sqrt{2}$.

The theory described by the action (4.5) has a three parameter family of classical solutions [12]:

$$T = \sqrt{2}\sinh^{-1}\left(\frac{\lambda}{\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}+\alpha\right)\right), \qquad A_x = \frac{1}{\sqrt{2}}\beta\,. \tag{4.6}$$

Here λ, α and β are the parameters labelling the solution with α and β being angular variables with periodicity 2π. Our goal is to find the metric in the three parameter moduli space labelled by λ, α and β, and compare this with the moduli space metric of the BCFT derived in section 3. For this we treat the parameters λ, α and β as collective coordinates, and consider a time dependent configuration of the form:

$$T(x,t) = \sqrt{2}\sinh^{-1}\left(\frac{\lambda(t)}{\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}+\alpha(t)\right)\right), \qquad A_x(x,t) = \frac{1}{\sqrt{2}}\beta(t)\,. \tag{4.7}$$

Substituting this into the action (4.5) we get the action of the collective coordinates. A short calculation gives:

$$\begin{aligned} S = -2\pi\sqrt{2}\,\mathcal{T}_1\Bigg[1 - \left(\frac{1}{\sqrt{1+\frac{1}{2}\lambda^2}} - \frac{1}{1+\frac{1}{2}\lambda^2}\right)\frac{\dot\lambda^2}{\lambda^2} - \left(1-\frac{1}{\sqrt{1+\frac{1}{2}\lambda^2}}\right)\dot\alpha^2 - \\ -\frac{1}{4}\frac{1}{\sqrt{1+\frac{1}{2}\lambda^2}}\dot\beta^2\Bigg] + \cdots\,, \end{aligned} \tag{4.8}$$

where $\ldots$ denotes terms involving four or more time derivatives. This gives the moduli space metric to be

$$ds^2 = 2\left[\left(\frac{1}{\sqrt{1+\frac{1}{2}\lambda^2}} - \frac{1}{1+\frac{1}{2}\lambda^2}\right)\frac{d\lambda^2}{\lambda^2} + \left(1-\frac{1}{\sqrt{1+\frac{1}{2}\lambda^2}}\right)d\alpha^2 + \frac{1}{4}\frac{1}{\sqrt{1+\frac{1}{2}\lambda^2}}d\beta^2\right]. \tag{4.9}$$

In order to compare this with the metric (3.8) found from direct BCFT analysis, we need to find the relation between the parameters (λ,α,β) appearing in (4.6) and the parameters (θ,ϕ,χ) labelling the BCFT. For this we note that for small λ, the solution given in (4.6) reduces to the configuration (3.15). Thus for small λ we can use the results of eqs. (3.19) to get

$$\alpha = \phi\,, \qquad \beta = 2\chi\,, \qquad \lambda \simeq 2\theta\,. \tag{4.10}$$

We expect that for the angular variables χ, ϕ, α and β this identification will continue to hold even for finite λ since in the effective field theory (conformal field theory) α (ϕ) has the interpretation of translation along S^1 and $\beta/2$ (χ) has the interpretation of switching on constant gauge field. The relation between λ and θ however is likely to be modified for finite λ. To determine this relation we use the same principle as was used in section 3, – namely compare the T_{xx} associated with these solutions. For the BCFT we have

$$T_{xx} = -\mathcal{T}_1\cos^2(\pi\widetilde{\lambda}) = -\mathcal{T}_1\cos^2\theta\,, \tag{4.11}$$

whereas for the solution (4.6) and the action (4.5)

$$T_{xx} = -\frac{V(T)}{\sqrt{1+(T')^2}} = -\frac{\mathcal{T}_1}{\sqrt{1+\frac{1}{2}\lambda^2}}\,. \tag{4.12}$$

Comparing (4.11) and (4.12) we get:

$$\frac{1}{1+\frac{1}{2}\lambda^2} = \cos^4\theta\,. \tag{4.13}$$

Thus the complete relation between the parameters (λ, α, β) and (θ, ϕ, χ) is given by:

$$\lambda = \sqrt{2}\,\sec^2\theta\sin\theta\sqrt{1+\cos^2\theta}\,, \qquad \alpha = \phi\,, \qquad \beta = 2\chi\,. \tag{4.14}$$

Note that as λ varies between 0 and ∞, θ varies between 0 and $\pi/2$. On the other hand since α and β are angular variables with periodicity 2π, ϕ and χ are angular variables with periodicities 2π and π respectively. Thus the ranges of (θ, ϕ, χ) are given by:

$$0 \le \theta \le \frac{\pi}{2}\,, \qquad 0 \le \phi \le 2\pi\,, \qquad 0 \le \chi \le \pi\,. \tag{4.15}$$

This is identical to (3.11) obtained by analysis of the moduli space of the BCFT.[3]

Using (4.14) we can express the metric (4.9) in terms of the variables (θ, χ, ϕ). The result is:

$$ds^2 = 2\left[\frac{4d\theta^2}{(1+\cos^2\theta)^2} + \sin^2\theta d\phi^2 + \cos^2\theta d\chi^2\right], \tag{4.16}$$

This is to be compared with the exact answer (3.8). We note first of all that the two metrics agree near $\theta = 0$. This is not surprising since the effective action (4.5) agrees with the string field theory action (3.12) for small T and A_x, and (3.12) does describe correctly the dynamics of small fluctuations on a non-BPS D-brane. However the similarity between (3.8) and (4.16) goes beyond the small θ region. The coefficients of $d\phi^2$ and $d\chi^2$ terms are reproduced exactly. Furthermore the ranges of (θ, ϕ, χ) labelling the solutions of the effective field theory and BCFT's are identical. The only difference between the two metrics is in the coefficient of the $d\theta^2$ term. For any given θ the coefficient of the $d\theta^2$ term is larger in the metric (4.16) derived from the effective field theory than in the exact metric (3.8). Thus the moduli space of the solutions of the effective field theory can be regarded as an elongated version of the exact moduli space S^3/Z_2 along the θ coordinate.

To get more insight into the geometric stucture of the moduli space described by the metric (4.16) we go to the double cover by taking χ to be a periodic variable with periodicity 2π. We can now consider a family of metrics interpolating between the metric (4.16) and the metric on S^3:

$$ds^2 = 2\left[\frac{4d\theta^2}{(2-u\sin^2\theta)^2} + \sin^2\theta d\phi^2 + \cos^2\theta d\chi^2\right]. \tag{4.17}$$

[3] The fact that the ranges of ϕ and χ agree between effective field theory and boundary conformal field theory is not a surprise since the relative normalization between ϕ and α (χ and β) was fixed by matching their periodicities. Matching of the range of θ however was not guaranteed.

As u varies from 0 to 1, we go from the metric on S^3 to the metric (4.16). For a given θ the coefficient of $d\theta^2$ increases monotonically as u increases. Thus as u varies from 0 to 1, (4.17) describes a family of metrics in which S^3 is elongated continuously along the θ direction.

Near $\theta = \pi/2$ the metric (4.17) looks like

$$ds^2 \simeq 2\left[d\phi^2 + \frac{4}{(2-u)^2}d\psi^2 + \psi^2 d\chi^2\right], \tag{4.18}$$

where $\psi = \frac{\pi}{2} - \theta$. This metric has a conical defect with deficit angle $u\pi$ at $\theta = \pi$. Thus in the process of elongating S^3 along the θ direction, the metric (4.17) also develops a conical singularity at $\theta = \pi/2$ where the excess curvature is stored. For the metric (4.16) the deficit angle at $\theta = \pi/2$ reaches the value π, i.e. we have a Z_2 orbifold singularity at this point. The actual moduli space is obtained by taking a further quotient of this manifold by Z_2, which generates a Z_2 orbifold singularity on the original S^3 and converts the Z_2 orbifold singularity of the metric (4.16) into a Z_4 orbifold singularity.

At $\theta = \pi/2$ ($\lambda = \infty$) the solution (4.6) represents infinitely sharp kink antikink pair located at $x = \sqrt{2}(\frac{\pi}{2} - \alpha)$ and $x = \sqrt{2}(\frac{3\pi}{2} - \alpha)$. Thus at this point a new branch of moduli space opens up, consisting of configurations where the kink and the antikink are placed at arbitrary points on the circle. The moduli space of these solutions, labelled by the locations of the kink and the antikink on S^1, clearly has the structure of a square torus $S^1 \times S^1$, and is joined to the other branch along the diagonal where the kink and the anti-kink are placed at diametrically opposite points. This is precisely what happens for the exact moduli space obtained from the BCFT analysis. In fact, in this case one can show that the full effective action living on the world-volume of the kink-antikink pair is given by the Dirac-Born-Infeld action on a D0-$\bar{\text{D}}$0 brane pair [14]. Thus the correspondence between the BCFT description and the world-volume description of the action goes far beyond the moduli space approximation.

In order to have a new branch opening up at the $\theta = \pi/2$ point, there must be additional massless modes corresponding to changing the relative separation between the kink and the antikink. Thus we should expect that when we analyze the spectrum of fluctuations around the background (4.6), a new massless mode, besides the three collective coordinates, must appear in the $\lambda \to \infty$ limit. In section 5 we verify this explicitly.

To summarize, we see that qualitatively the structure of the moduli space of classical solutions in the effective field theory described by the action (4.1) is in agreement with the exact result obtained in section 3 using the bondary conformal field theory analysis. This shows that the effective field theory (4.1) is a good candidate for providing a qualitative description of tachyon dynamics on an unstable D-brane.

5. Origin of the additional massless mode at the Dirichlet point

In the analysis of sections 3 and 4 we have seen that at the point $\theta = \pi/2$, additional massless modes appear, and a new branch of the moduli space opens up. The origin of these additional massless modes in the BCFT is well understood [3, 4] — the usual zero momentum tachyonic mode and various other massive modes at the $\theta = 0$ point become massless at $\theta = \pi/2$ and give rise to the new branch of the moduli space. In this section we shall explore the origin of the additional massless mode in the effective field theory description of section 4. In particular we shall show that in the effective field theory also the zero momentum tachyonic mode at the $\theta = 0$ point flows into a massless mode at the $\theta = \pi/2$ point, and gives rise to the additional direction in the moduli space that allows us to change the relative separation between the kink antikink pair.

We shall begin by analyzing the small oscillation modes around the classical solution (4.6). Due to the shift symmetry $A_x \to A_x + c$, and the translational invariance along x of the effective field theory action (4.5), we can work at the point $\alpha = \beta = 0$ without any loss of generality. Thus the solution is:

$$T = \sqrt{2}\sinh^{-1}\left(\frac{\lambda}{\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)\right), \qquad A_x = 0\,. \tag{5.1}$$

In carrying out the analysis of fluctuation around this solution, we shall ignore the fluctuation of the gauge field A_x since, as we shall see, the additional massless mode at $\theta = \pi/2$ appears from the tachyon field T. This is a consistent truncation due to $A_x \to -A_x$ symmetry of the action. Thus we focus our attention on the following type of fluctuation around the classical solution (5.1)

$$T(x,t) = \sqrt{2}\sinh^{-1}\left(\frac{\lambda}{\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)\right) + \chi(x,t)\,, \qquad A_x = 0\,. \tag{5.2}$$

Substituting this into the effective action (4.5), and ignoring the constant term, we get the following action for the mode χ to quadratic order in χ:

$$\mathcal{S} = \frac{\mathcal{T}_1}{\sqrt{4+2\lambda^2}}\int dt \int_0^{2\pi\sqrt{2}} dx\left[\dot{\chi}^2 - \frac{2+\lambda^2\cos^2(x/\sqrt{2})}{2+\lambda^2}(\chi')^2 + \frac{1}{2+\lambda^2\cos^2(x/\sqrt{2})}\chi^2\right] + \\ + \mathcal{O}(\chi^3)\,. \tag{5.3}$$

Thus the linearized equation of motion for χ is:

$$-\frac{d}{dx}\left[\frac{2+\lambda^2\cos^2(x/\sqrt{2})}{2+\lambda^2}\frac{d\chi}{dx}\right] - \frac{1}{2+\lambda^2\cos^2(x/\sqrt{2})}\chi = -\ddot{\chi}\,. \tag{5.4}$$

A mode of χ of mass m will correspond to a time dependent solution of the form $\chi_m(x)e^{imt}$. Substituting this into (5.4) we get:

$$-\frac{d}{dx}\left[\frac{2+\lambda^2\cos^2(x/\sqrt{2})}{2+\lambda^2}\frac{d\chi_m}{dx}\right] - \frac{1}{2+\lambda^2\cos^2(x/\sqrt{2})}\chi_m = m^2\chi_m\,. \tag{5.5}$$

Using the relation between λ and the parameter θ given in (4.13) we can rewrite this equation as:

$$-\frac{d}{dx}\left[\left(\cos^4\theta + \sin^2\theta(1+\cos^2\theta)\cos^2\frac{x}{\sqrt{2}}\right)\frac{d\chi_m}{dx}\right] - \\ -\frac{\cos^4\theta}{2\left(\cos^4\theta + \sin^2\theta(1+\cos^2\theta)\cos^2(x/\sqrt{2})\right)}\chi_m = m^2\chi_m\,. \tag{5.6}$$

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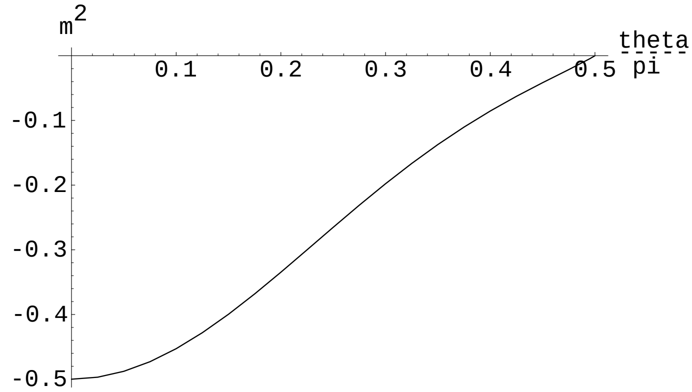

Figure 1: Numerical results for the mass^2 of the lowest mode of the tachyon as a function of the parameter θ/π labelling the classical solution in the effective field theory. Note that the mass^2 goes from $-\frac{1}{2}$ to 0 as the parameter θ varies from 0 to $\pi/2$.

This is an eigenvalue equation for the operator

$$-\frac{d}{dx}\left[\left(\cos^4\theta+\sin^2\theta(1+\cos^2\theta)\cos^2\frac{x}{\sqrt{2}}\right)\frac{d}{dx}\right]-\frac{\cos^4\theta}{2(\cos^4\theta+\sin^2\theta(1+\cos^2\theta)\cos^2(x/\sqrt{2}))} \tag{5.7}$$

with the eigenvalues giving the mass^2 spectrum of the $(0+1)$ dimensional theory living on this classical solution. For any value of θ this operator contains a zero eigenvalue, with the corresponding eigenfunction χ_0 denoting deformation induced by the parameter α in eq. (4.6):

$$\chi_0(x)\propto\frac{d}{dx}\left[\sinh^{-1}\left(\frac{\lambda}{\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)\right)\right]\propto\frac{\sin(x/\sqrt{2})}{\sqrt{1+\tan^2\theta(1+\sec^2\theta)\cos^2(x/\sqrt{2})}}\,. \tag{5.8}$$

It can be checked explicitly that χ_0 defined in (5.8) satisfies (5.6) with $m=0$.

From the form of χ_0 given in (5.8) we see that it has two nodes in the range $0\le x<2\pi\sqrt{2}$ — at $x=0$ and at $x=\pi\sqrt{2}$. Arguments given in [17] then show that there must be another eigenfunction of the operator (5.7) of lower m^2 eigenvalue, *i.e.* with $m^2\le 0$. Such an eigenfunction clearly exists at $\theta=0$ — it is simply the constant mode and has $m^2=-1/2$. For $\theta\neq 0$, the eigenfunction and the eigenvalue can be computed numerically. In figure 1 we show the numerical results for the lowest m^2 eigenvalue as a function of θ/π. As is clear from this figure, as θ varies from 0 to $\pi/2$, the lowest m^2 value varies smoothly from $-1/2$ to 0. This is consistent with the general argument of [17] that the tachyonic mode must disappear in the $\lambda\to\infty$ (i.e. $\theta\to\pi/2$) limit.

In order to find the physical interpretation of this mode in the $\theta\to\pi/2$ limit, we can examine the form of the eigenfunction. In figure 2 we have plotted the suitably normalized eigenfunction as a function of $(x/2\pi\sqrt{2})$ for $\theta=.45\pi$. For comparison, we have also plotted the translational mode given in (5.8) on the same graph. Comparing the two graphs we see that both are peaked around the points $x=\pi/\sqrt{2}$, and $3\pi/\sqrt{2}$, i.e. the points where the kink and the antikink are located in the $\theta\to\pi/2$ limit. Furthermore the

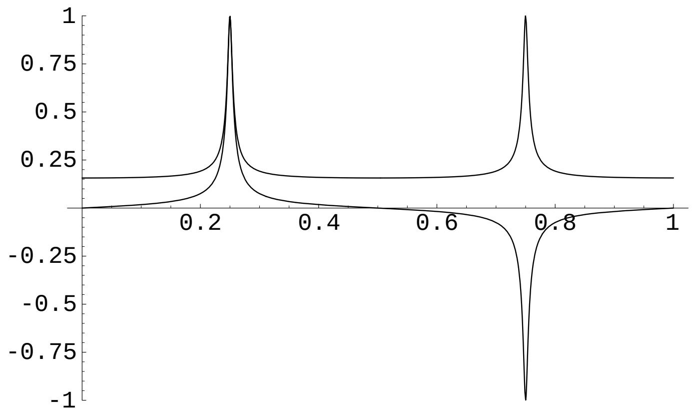

Figure 2: Plot of χ vs. $x/(2\pi\sqrt{2})$ for the two lowest modes for $\theta = .45\pi$. The lowest mode has positive peaks at $x = \pi/2$ and at $x = 3\pi/2$, whereas the first excited mode (translational zero mode) has a positive peak at $x = \pi/2$ and a negative peak at $x = 3\pi/2$.

forms of the lowest mode and the translational mode agree closely near $x = \pi/\sqrt{2}$, but differ by a sign near $x = 3\pi/\sqrt{2}$. Numerical results also show that for both eigenmodes the peaks become sharper as θ approaches $\pi/2$. Since at $\theta = \pi/2$ the translational mode corresponds to translating the kink and the antikink in the same direction, keeping their relative separation intact, this suggests that the lowest mode corresponds to translating the kink and the antikink in the opposite direction. This is precisely the mode that takes us to the other branch of the moduli space.

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